The Laplace Transform is frequently encountered in mathematics, physics, engineering and other fields. However, the spectral properties of the Laplace Transform tend to complicate its numerical treatment; therefore, the closely related "Truncated" Laplace Transforms are often used in applications.
We have constructed efficient algorithms for the evaluation of the left singular functions and singular values of the Truncated Laplace Transform. Together with the previously introduced algorithms for the evaluation of the right singular functions, these algorithms provide the Singular Value Decomposition of the Truncated Laplace Transform.
The resulting algorithms are applicable to all environments likely to be encountered in applications, including the evaluation of singular functions corresponding to extremely small singular values (e.g. 10^{-1000}).

On the Analytical and Numerical Properties of the Truncated Laplace Transform II.

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Technical Report YALEU/DCS/TR-1507
May 29, 2015

[†] This author's research was supported in part by the ONR grants #N00014-11-1-0718, #N00014-10-1-0570 and #N00014-14-1-0797 and the NSF grant #1309858.
[‡] This author's research was supported in part by the ONR grants #N00014-10-1-0570 and #N00014-14-1-0797.

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Keywords: *Truncated Laplace Transform, SVD.*

1 Introduction

The Laplace Transform $\mathcal{L}$ is a linear mapping $L^2[0,\infty) \to L^2[0,\infty)$; for a function $f \in L^2[0,\infty)$, it is defined by the formula

$$(\mathcal{L}(f))(\omega) = \int_0^\infty e^{-t\omega} f(t)\mathrm{d}t. \tag{1}$$

As is well-known, $\mathcal{L}$ has a continuous spectrum, and $\mathcal{L}^{-1}$ is not continuous (see, for example, [3]). These and related properties tend to complicate the numerical treatment of $\mathcal{L}$.

In addressing these problems, we find it useful to draw an analogy between the numerical treatment of the Laplace Transform, and the numerical treatment of the Fourier Transform $\mathcal{F}$; for a function $f \in L^1(\mathbb{R})$, the latter is defined by the formula

$$(\mathcal{F}(f))(\omega) = \int_{-\infty}^\infty e^{-it\omega} f(t)\mathrm{d}t, \tag{2}$$

where $\omega \in \mathbb{R}$.

In various applications in mathematics and engineering, it is useful to define the "Truncated" Fourier Transform $\mathcal{F}_c : L^2[-1,1] \to L^2[-1,1]$; for a given $c > 0$, $\mathcal{F}_c$ of a function $f \in L^2[-1,1]$ is defined by the formula

$$(\mathcal{F}_c(f))(\omega) = \int_{-1}^1 e^{-ict\omega} f(t)\mathrm{d}t. \tag{3}$$

The operator $\mathcal{F}_c$ has been analyzed extensively; one of most notable observations, made by Slepian and Pollak around 1960, was that the integral operator $\mathcal{F}_c$ commutes with a second order differential operator (see [18]). This property of $\mathcal{F}_c$ was used in analytical and numerical investigations of the eigendecomposition of this operator; for example in [18, 9, 10, 16, 17, 19, 15].

For $0 < a < b < \infty$, the linear mapping $\mathcal{L}_{a,b} : L^2[a,b] \to L^2[0,\infty)$ defined by the formula

$$(\mathcal{L}_{a,b}(f))(\omega) = \int_a^b e^{-t\omega} f(t)\mathrm{d}t, \tag{4}$$

will be referred to as the *Truncated Laplace Transform* of f; obviously, $\mathcal{L}_{a,b}$ is a compact operator (see, for example, [3]) .

The Singular Value Decomposition (SVD) of $\mathcal{L}_{a,b}$ has been analyzed, inter alia, in [3] and [5]; Bertero and Grünbaum observed that each of the symmetric operators $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$ and $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*$ commutes with a differential operator (see [5]). Despite [5, 6, 3, 2, 4, 13, 8, 7], much more is known about the numerical and analytical properties of $\mathcal{F}_c$ than about the properties of $\mathcal{L}_{a,b}$.

We have constructed algorithms for the efficient evaluation of the SVD of $\mathcal{L}_{a,b}$. In [12] we introduced algorithms for the efficient evaluation of the right singular functions and singular values of $\mathcal{L}_{a,b}$. In this paper, we introduce an efficient algorithm for the numerical evaluation

of the left singular functions, and an additional algorithm for computing the singular value. Additional properties of the Truncated Laplace transform will be discussed in upcoming papers.

This paper is organized as follows. Section §2 summarizes the various standard mathematical facts and simple derivations that are used later in this paper, as well as various properties of the Truncated Laplace Transform. Section §3 contains the derivation of various properties of the truncated Laplace transform which are related to the left singular functions; the properties are used in the algorithms presented in the following section. Section §4 describes the algorithms for the evaluation of the left singular functions and singular values of the Truncated Laplace Transform. Section §5 contains numerical results obtained using the algorithms. Section §6 contains generalizations and conclusions.

2 Preliminaries

2.1 The Laguerre Functions

In this section we summarize some of the properties of the Laguerre Polynomials, Generalized Laguerre polynomials, and Laguerre Functions. The Laguerre polynomial $L_k(x)$ of degree $k \geq 0$ is defined by the formula

$$L_k(x) = \sum_{m=0}^{k} (-1)^m \binom{k}{k-m} \frac{1}{m!} x^m. \tag{5}$$

The Generalized Laguerre Polynomial of order $\alpha > -1$ and degree $k \geq 0$, denoted by $L_k^{(\alpha)}(x)$, is defined by the formula

$$L_k^{(\alpha)}(x) = \sum_{m=0}^{k} (-1)^m \binom{k+\alpha}{k-m} \frac{1}{m!} x^m. \tag{6}$$

The Laguerre Function of degree $k \geq 0$, which we denote by Φ_k, is defined via the formula

$$\Phi_k(x) = e^{-x/2} L_k(x). \tag{7}$$

The following well known properties of the Laguerre Polynomials and the Generalized Laguerre Polynomials can be found, inter alia, in [1].

$$L_k^{\alpha-1}(x) = L_k^{\alpha}(x) - L_{k-1}^{\alpha}(x) \tag{8}$$

$$\frac{d}{\mathrm{d}x} L_k(x) = -L_{k-1}^{(1)} \tag{9}$$

$$xL_k(x) = -(k+1)L_{k+1}(x) + (2k+1)L_k(x) - kL_{k-1}(x) \tag{10}$$

$$\int_0^\infty e^{-xt} L_k(x) \mathrm{d}x = (t-1)^k t^{-k-1} \tag{11}$$

As is well-known, the Laguerre Polynomials $L_0(x), L_1(x), ...$ form an orthonormal basis in the Hilbert space induced by the weighted inner product

$$(f, g) = \int_0^\infty e^{-x} f(x) g(x) \mathrm{d}x. \tag{12}$$

It follows that the Laguerre functions $\Phi_0(x), \Phi_1(x), ...$ form an orthonormal basis in the standard $L^2[0, \infty)$ sense.

By (11) and (7),

$$\int_0^\infty e^{-xt} \Phi_k(x) \mathrm{d}x = \left(t - \frac{1}{2}\right)^k \left(t + \frac{1}{2}\right)^{-k-1}. \tag{13}$$

2.2 The Legendre Polynomials

In this subsection we summarize some of the properties of the Normalized Shifted Legendre Polynomials, derived from the standard Legendre Polynomials. We define the *Normalized Shifted Legendre Polynomial* of degree $k = 0, 1, ...$, which we will be denoting by $\overline{P_k^*}$, by the formula

$$\overline{P_k^*}(x) = P_k(2x-1)\sqrt{2k+1}; \tag{14}$$

where P_k is the Legendre Polynomial of degree k; the standard definition of the Legendre Polynomials can be found, inter alia, in [1].

As is well-known, the Legendre Polynomials form an orthogonal basis in $L^2[-1, 1]$, but they are not normalized; a simple calculation shows that the Normalized Shifted Legendre Polynomials $\overline{P_0^*}, \overline{P_1^*}, ...$ form an orthonormal basis in $L^2[0, 1]$.

By (14), the first Normalized Shifted Legendre Polynomial is a constant,

$$\overline{P_0^*}(x) = 1. \tag{15}$$

2.3 Singular Value Decomposition of integral operators

The SVD of integral operators and its key properties are summarized in the following theorem, which can be found, for example, in [20].

Theorem 2.1. *Suppose that the function $K : [c, d] \times [a, b] \to \mathbb{R}$ is square integrable, and $T : L^2[a, b] \to L^2[c, d]$ is defined by the formula*

$$(T(f))(x) = \int_a^b K(x, t) f(t) \mathrm{d}t. \tag{16}$$

Then, there exist two orthonormal sequences of functions $u_0, u_1, ...,$ where $u_n : [a,b] \to \mathbb{R}$ and $v_0, v_1, ...,$ where $v_n : [c,d] \to \mathbb{R}$, and a sequence $\alpha_0, \alpha_1, ... \in \mathbb{R}$, where $\alpha_0 \geq \alpha_1 \geq ... \geq 0$, such that

$$(T(f))(x) = \sum_{n=0}^{\infty} \alpha_n \left(\int_a^b u_n(t) f(t) \mathrm{d}t \right) v_n(x) \tag{17}$$

for any $f \in L^2[a,b]$. The sequence $\alpha_0, \alpha_1, ...$ is uniquely determined by K.

The functions $u_0, u_1, ...$ are referred to as the *right singular functions*, the functions $v_0, v_1, ...$ are referred to as the *left singular functions*, and the values $\alpha_0, \alpha_1, ...$ are referred to as the *singular values* of the operator T. Together, the right singular functions, the left singular functions and the singular values are referred to as the SVD of the operator T.

It immediately follows from Theorem 2.1 that

$$T(u_n) = \alpha_n v_n, \tag{18}$$

$$T^*(v_n) = \alpha_n u_n. \tag{19}$$

Observation 2.2. The right singular functions $u_0, u_1, ..$ of T are eigenfunctions of the operator $T^* \circ T$ and the left singular functions $v_0, v_1, ...$ are eigenfunctions of the operator $T \circ T^*$; the singular values $\alpha_0, \alpha_1, ...$ of T are the square roots of the eigenvalues of $T^* \circ T$ and $T \circ T^*$. In other words, for every $n = 0, 1, ...,$

$$((T^* \circ T)(u_n))(\tau) = \int_c^d \overline{K(x,\tau)} \left(\int_a^b K(x,t) u_n(t) \mathrm{d}t \right) \mathrm{d}x = \alpha_n^2 u_n(\tau) \tag{20}$$

and

$$((T \circ T^*)(v_n))(\xi) = \int_a^b K(\xi,t) \left(\int_c^d \overline{K(x,t)} v_n(x) \mathrm{d}x \right) \mathrm{d}t = \alpha_n^2 v_n(\xi) \tag{21}$$

Remark 2.3. The function K can be expressed using the singular functions as follows (see [20]),

$$K(x,t) = \sum_{n=0}^{\infty} v_n(x) \alpha_n u_n(t) \tag{22}$$

and it can be approximated by truncation of small singular values (also see [20]):

$$K(x,t) \simeq \sum_{n=0}^{p} v_n(x) \alpha_n u_n(t) \tag{23}$$

2.4 The Truncated Laplace Transform

Definition 2.4. *For any pair of real numbers a, b, such that $0 < a < b < \infty$, the* Truncated Laplace Transform $\mathcal{L}_{a,b}$ *is the linear mapping $L^2[a,b] \to L^2[0,\infty)$, defined by the formula*

$$\left(\mathcal{L}_{a,b}(f)\right)(\omega) = \int_a^b e^{-t\omega} f(t)\mathrm{d}t, \tag{24}$$

Obviously, the adjoint of $\mathcal{L}_{a,b}$ is

$$\left((\mathcal{L}_{a,b})^*(g)\right)(t) = \int_0^\infty e^{-t\omega} g(\omega)\mathrm{d}\omega. \tag{25}$$

The operators $\mathcal{L}_{a,b}$ and $(\mathcal{L}_{a,b})^*$ are compact, the range of $(\mathcal{L}_{a,b})^*$ is dense in $L^2[a,b]$ and the range of $\mathcal{L}_{a,b}$ is dense in $L^2[0,\infty)$ (see, for example, [3]).

2.5 The SVD of the Truncated Laplace Transform

By Theorem 2.1, there exist an orthonormal sequence of right singular functions $u_0, u_1, \ldots \in L^2[a,b]$, an orthonormal sequence of left singular functions $v_0, v_1, \ldots \in L^2[0,\infty)$ and a sequence of real numbers $\alpha_0, \alpha_1, \ldots \in \mathbb{R}$ such that

$$\left(\mathcal{L}_{a,b}(f)\right)(\omega) = \sum_{n=0}^{\infty} \alpha_n \left(\int_a^b u_n(t) f(t)\mathrm{d}t\right) v_n(\omega), \tag{26}$$

and for all $n = 0, 1, \ldots,$

$$\mathcal{L}_{a,b}(u_n) = \alpha_n v_n, \tag{27}$$

$$(\mathcal{L}_{a,b})^*(v_n) = \alpha_n u_n, \tag{28}$$

and

$$\alpha_n \geq \alpha_{n+1} \geq 0. \tag{29}$$

Remark 2.5. The multiplicity of the singular values of $\mathcal{L}_{a,b}$ is one (see [5]); in other words, for all $n = 0, 1, \ldots$

$$\alpha_n > \alpha_{n+1}. \tag{30}$$

Remark 2.6. According to Observation 2.2, the right singular functions $u_0, u_1, \ldots$ of $\mathcal{L}_{a,b}$ are eigenfunctions of the integral operator $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b} : L^2[a,b] \to L^2[a,b]$ given by the formula

$$\left(\left((\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}\right)(f)\right)(t) = \int_0^\infty e^{-\omega t} \left(\int_a^b e^{-\omega s} f(s)\mathrm{d}s\right) \mathrm{d}\omega = \int_a^b \frac{1}{t+s} f(s)\mathrm{d}s,$$

$$ \tag{31} $$

and the corresponding eigenvalues of $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$ are $\alpha_0^2, \alpha_1^2, ...$, where α_n is the singular value of $\mathcal{L}_{a,b}$ associated with the right singular function u_n. In other words,

$$ \left(\left((\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}\right)(u_n)\right)(t) = \int_a^b \frac{1}{t+s} u_n(s)\mathrm{d}s = \alpha_n^2 u_n(t). \tag{32} $$

Similarly, the left singular functions v_n of $\mathcal{L}_{a,b}$ are eigenfunctions of the integral operator $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^* : L^2[0,\infty) \to L^2[0,\infty)$ given by the formula

$$ \begin{gathered} \left(\left(\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*\right)(g)\right)(\omega) = \\ = \int_a^b e^{-\omega t} \left(\int_0^\infty e^{-\rho t} g(\rho)\mathrm{d}\rho\right)\mathrm{d}t = \int_0^\infty \frac{e^{-a(\omega+\rho)} - e^{-b(\omega+\rho)}}{\omega+\rho} g(\rho)\mathrm{d}\rho, \end{gathered} \tag{33} $$

and the corresponding eigenvalues $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*$ are $\alpha_0^2, \alpha_1^2, \ldots$. In other words,

$$ \left(\left(\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*\right)(v_n)\right)(\omega) = \int_0^\infty \frac{e^{-a(\omega+\rho)} - e^{-b(\omega+\rho)}}{\omega+\rho} v_n(\rho)\mathrm{d}\rho = \alpha_n^2 v_n(\omega). \tag{34} $$

2.6 The differential operators $\tilde{D}_t$ and $\hat{D}_\omega$ associated with the singular functions of $\mathcal{L}_{a,b}$

In this subsection we summarize several properties related to the differential operator $\tilde{D}_t$, defined by the formula

$$ \left(\tilde{D}_t(f)\right)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left((t^2-a^2)(b^2-t^2)\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right) - 2(t^2-a^2)f(t), \tag{35} $$

where $f \in C^2[a,b]$; and properties related to the differential operator $\hat{D}_\omega$, defined by the formula

$$ \begin{gathered} \left(\hat{D}_\omega(f)\right)(\omega) = \\ = -\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left(\omega^2 \frac{\mathrm{d}^2}{\mathrm{d}\omega^2} f(\omega)\right) + (a^2+b^2)\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\omega^2\frac{\mathrm{d}}{\mathrm{d}\omega}f(\omega)\right) + \left(-a^2b^2\omega^2 + 2a^2\right)f(\omega), \end{gathered} \tag{36} $$

where $f \in C^4[0,\infty) \cap L^2[0,\infty)$. For a derivation of these properties, see [5].

Theorem 2.7. *The differential operator $\tilde{D}_t$, defined in (35), commutes with the integral operator $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$, (specified in (31)) in $L^2[a,b]$. In other words,*

$$ \tilde{D}_t \circ \left((\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}\right) = \left((\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}\right) \circ \tilde{D}_t \tag{37} $$

Theorem 2.8. *The differential operator $\hat{D}_\omega$, defined in (36), commutes with the integral operator $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*$, (specified in (33)) in $L^2[0,\infty)$. In other words,*

$$\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^* \circ \hat{D}_\omega = \hat{D}_\omega \circ \mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*. \tag{38}$$

Theorem 2.9. *The right singular functions $u_0, u_1, ...$ (defined in (27)) of $\mathcal{L}_{a,b}$ (defined in (24)) are also the eigenfunctions of $\tilde{D}_t$.*

Theorem 2.10. *The left singular functions $v_0, v_1, ...$ (defined in (27)) of $\mathcal{L}_{a,b}$ (defined in (24)) are also the eigenfunctions of $\hat{D}_\omega$.*

We denote the eigenvalues of the differential operator $\tilde{D}_t$ by $\tilde{\chi}_0, \tilde{\chi}_1, ...$, and the eigenvalues of the differential operator $\hat{D}_\omega$ by $\chi_0^*, \chi_1^*, ...$ By Theorem 2.9, the singular function u_n is the solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((t^2-a^2)(b^2-t^2)\frac{\mathrm{d}}{\mathrm{d}t}u_n(t)\right) - 2(t^2-a^2)u_n(t) = \tilde{\chi}_n u_n(t), \tag{39}$$

and by Theorem 2.10, the left singular function v_n is the solution to the differential equation

$$\begin{aligned} &-\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left(\omega^2\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}v_k(\omega)\right) + (a^2+b^2)\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\omega^2\frac{\mathrm{d}}{\mathrm{d}\omega}v_k(\omega)\right) + (-a^2b^2\omega^2+2a^2)v_k(\omega) = \\ &= \chi_k^* v_k(\omega). \end{aligned} \tag{40}$$

Remark 2.11. The singular values α_n (defined in (27)) of the integral operator $\mathcal{L}_{a,b}$ are known to decay exponentially as n grows; consequently, the direct numerical computation of the singular functions of $\mathcal{L}_{a,b}$ beyond the first few singular functions is impossible.

The differential operators $\tilde{D}_t$ and $\hat{D}_\omega$ are advantageous in the numerical treatment of the singular functions u_n and v_n because their eigenvalues are well-separated.

2.7 Properties of the right singular functions u_n

In this section we presents some of the numerical properties of the right singular functions u_n, a more detailed discussion of these properties is found in [12].

We find it convenient to define the function $\psi_n(x)$ on the interval $[0,1]$ by the formula

$$\psi_n(x) = \sqrt{b-a}\; u_n(a+(b-a)x). \tag{41}$$

We introduce the notation $h^n = (h_0^n, h_1^n, ...)^\top$ for the vector of coefficients of the expansion of the function ψ_n in the basis of Normalized Shifted Legendre Polynomials (defined in (14)); where the element h_k^n is defined by the formula

$$h_k^n = \int_0^1 \psi_n(x)\overline{P_k^*}(x)\mathrm{d}x, \tag{42}$$

so that

$$\psi_n(x) = \sum_{k=0}^{\infty} h_k^n \overline{P_k^*}(x). \tag{43}$$

Theorem 2.12. *The vector of coefficients h^n is the $n+1$-th eigenvector of the five-diagonal matrix M, defined by the formula*

$$\begin{aligned}
M_{k-2,k} &= -\frac{(k-1)^2 k^2}{4\sqrt{2k-3}(2k-1)\sqrt{2k+1}}, \\
M_{k-1,k} &= -\frac{k^3(1+\beta)}{\sqrt{2k-1},\sqrt{2k+1}} \\
M_{k,k} &= -\frac{\left(-4-6\beta-2k\beta(2+3\beta)+k^2\left(7+12\beta+2\beta^2\right)+\left(2k^3+k^4\right)\left(7+16\beta+8\beta^2\right)\right)}{2(2k-1)(2k+3)}, \\
M_{k+1,k} &= -\frac{(k+1)^3(1+\beta)}{\sqrt{2k+1}\sqrt{2k+3}}, \\
M_{k+2,k} &= -\frac{(k+1)^2(k+2)^2}{4\sqrt{2k+1}(2k+3)\sqrt{2k+5}}.
\end{aligned} \tag{44}$$

where

$$\beta = \frac{2a}{b-a}. \tag{45}$$

3 Analytical apparatus

3.1 Expansion of v_n in the basis of Laguerre functions

Suppose that g is a smooth function in $L^2[0,\infty)$. Then, g can be expressed in the basis of Laguerre functions Φ_k (defined in (7)); let $\eta = (\eta_0, \eta_1, ...)^\top$ be the a vector where

$$\eta_k = \int_0^\infty g(\omega)\Phi_k(\omega)\mathrm{d}\omega, \tag{46}$$

then clearly η is the vector of coefficients in the expansion,

$$g(\omega) = \sum_{k=0}^{\infty} \eta_k \Phi_k(\omega). \tag{47}$$

We introduce the notation $\eta^n = (\eta_0^n, \eta_1^n, ...)^\top$ for the vector of coefficients of the expansion of the left singular function v_k (defined in (27)) in the basis of Laguerre functions; where the element η_k^n is defined by the formula

$$\eta_k^n = \int_0^\infty v_n(\omega)\Phi_k(\omega)\mathrm{d}\omega, \tag{48}$$

so that

$$v_n(\omega) = \sum_{k=0}^{\infty} \eta_k^n \Phi_k(\omega). \tag{49}$$

3.2 A matrix representation of the differential operator $\hat{D}_\omega$ in the basis of Φ_k

The purpose of this subsection is to express the differential operator $\hat{D}_\omega$ (defined in (36)) in the basis of Laguerre Functions Φ_k as the matrix $\hat{M}$ described in Lemma 3.1. Theorem 3.2 shows that the matrix $\hat{M}$ is in fact a five-diagonal matrix; and Corollary 3.3 provides the relationship between the eigenvectors of $\hat{M}$ and the left singular functions v_n defined in (27).

Lemma 3.1. *Let g be a smooth function with the expansion $\eta = (\eta_0, \eta_1, \ldots)^\top$ specified in (47):*

$$g(\omega) = \sum_{k=0}^{\infty} \eta_k \Phi_k(\omega). \tag{50}$$

Suppose that $\psi = \hat{D}_\omega(g)$, with the expansion $c = (c_0, c_1, \ldots)^\top$ such that

$$\psi(\omega) = \sum_{k=0}^{\infty} c_k \Phi_k(\omega). \tag{51}$$

Then,

$$c = \hat{M}\eta, \tag{52}$$

where the matrix elements $\hat{M}_{jk}$ of $\hat{M}$ are defined via the formula

$$\hat{M}_{jk} = \int_0^\infty \Phi_j(\omega) \left(\hat{D}_\omega\left(\Phi_k\right)\right)(\omega) \mathrm{d}\omega, \tag{53}$$

with $0 \le j, k < \infty$.

Proof. By the linearity of the differential operator $\hat{D}_\omega$ (defined in (36)),

$$\psi(\omega) = \left(\hat{D}_\omega(g)\right)(\omega) = \sum_{k=0}^{\infty} \eta_k \left(\hat{D}_\omega(\Phi_k)\right)(\omega). \tag{54}$$

Combining (51) and (54),

$$\sum_{k=0}^{\infty} c_k \Phi_k(\omega) = \sum_{k=0}^{\infty} \eta_k \left(\hat{D}_\omega(\Phi_k)\right)(\omega). \tag{55}$$

Now, by multiplying both sides of (55) by Φ_j and integrating, we have

$$c_j = \int_0^\infty \left(\sum_{k=0}^{\infty} \eta_k \left(\hat{D}_\omega(\Phi_k)\right)(\omega)\right) \Phi_j(\omega) \mathrm{d}\omega. \tag{56}$$

By linearity,

$$c_j = \sum_{k=0}^{\infty} \eta_k \left(\int_0^\infty \Phi_j(\omega) \left(\hat{D}_\omega(\Phi_k)\right)(\omega) \mathrm{d}\omega\right). \tag{57}$$

□

Theorem 3.2. *For any $k \geq 0$,*

$$\begin{aligned}
&\left(\hat{D}_\omega(\Phi_k)\right)(\omega) = \\
&- \frac{\left(4a^2-1\right)\left(4b^2-1\right)(k-1)k}{16}\Phi_{k-2}(\omega) \\
&+ \frac{k^2\left(16a^2b^2-1\right)}{4}\Phi_{k-1}(\omega) \\
&+ \frac{k(k+1)\left(-48a^2b^2-4a^2-4b^2-3\right)+\left(-16a^2b^2+12a^2-4b^2-1\right)}{8}\Phi_k(\omega) \\
&+ \frac{(k+1)^2\left(16a^2b^2-1\right)}{4}\Phi_{k+1}(\omega) \\
&- \frac{\left(4a^2-1\right)\left(4b^2-1\right)(k+2)(k+1)}{16}\Phi_{k+2}(\omega),
\end{aligned} \tag{58}$$

where Φ_k is the Laguerre function defined in (7).

In other words, $\hat{M}$ is the symmetric five-diagonal matrix with non-zero entries defined by the formulae

$$\begin{aligned}
\hat{M}_{k-2,k} &= -\frac{\left(4a^2-1\right)\left(4b^2-1\right)(k-1)k}{16}, \\
\hat{M}_{k-1,k} &= \frac{k^2\left(16a^2b^2-1\right)}{4}, \\
\hat{M}_{k,k} &= \frac{k(k+1)\left(-48a^2b^2-4a^2-4b^2-3\right)+\left(-16a^2b^2+12a^2-4b^2-1\right)}{8}, \\
\hat{M}_{k+1,k} &= \frac{(k+1)^2\left(16a^2b^2-1\right)}{4}, \\
\hat{M}_{k+2,k} &= -\frac{\left(4a^2-1\right)\left(4b^2-1\right)(k+2)(k+1)}{16}.
\end{aligned} \tag{59}$$

Proof. By the definition of $\hat{D}_\omega$ in (36),

$$\begin{aligned}
&\left(\hat{D}_\omega(\Phi_k)\right)(x) = \\
&= -\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\omega^2\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\Phi_k(\omega) + (a^2+b^2)\frac{\mathrm{d}}{\mathrm{d}\omega}\omega^2\frac{\mathrm{d}}{\mathrm{d}\omega}\Phi_k(x) + (-a^2b^2\omega^2+2a^2)\Phi_k(\omega)
\end{aligned} \tag{60}$$

A somewhat tedious derivation from (60), using identities (8), (9) and (10), yields (58). □

Corollary 3.3. *Suppose that $\eta^n = (\eta_0^n, \eta_1^n, ...)^\top$ is the vector of coefficients defined in (48), in the expansion of the left singular function v_n defined in (27); then, η^n is the $n+1$-th eigenvector of $\hat{M}$:*

$$\hat{M}\eta^n = \chi_n^* \eta^n, \tag{61}$$

where $\hat{M}$ is the five-diagonal matrix (59), χ_n^* are the eigenvalues of the differential operator $\hat{D}_\omega$, and $k = 0, 1, 2, ...$

Proof. By (40), v_n is an eigenfunction of $\hat{D}_\omega$, with the eigenvalue χ_n^*:

$$\left(\hat{D}_\omega v_k\right)(\omega) = \chi_k^* v_k(\omega), \tag{62}$$

so that

$$\left(\hat{D}_\omega \left(\sum_{k=0}^{\infty} \eta_k^n \Phi_k\right)\right)(\omega) = \left(\hat{D}_\omega v_k\right)(\omega) = \chi_k^* \sum_{k=0}^{\infty} \eta_k^n \Phi_k(\omega). \tag{63}$$

Therefore, by Lemma 3.1 we obtain (3.3).

□

Remark 3.4. If the operator $\hat{D}_\omega$ is represented in the basis associated with Hermite polynomials, rather than in the basis of Laguerre functions, the resulting matrix $\hat{M}$ in Lemma 3.1 and Theorem 3.2 is seven-diagonal and of certain analytical advantage. This approach is under investigation and will be reported at a later date.

3.2.1 The special cases $\mathcal{L}_{1/2,\gamma/2}$ and $\mathcal{L}_{1/2\gamma,1/2}$

An inspection of formula (59) immediately indicates that there are two special cases in which the second off-diagonal of the matrix $\hat{M}$ defined in (59) vanishes, and $\hat{M}$ becomes a tridiagonal band matrix.

The first case occurs when $a = 1/2$. The substitution of $a = 1/2, b = \gamma/2$ into (59) yields

$$\begin{aligned} \hat{M}_{k-1,k} =& \frac{1}{4}\left(\gamma^2 - 1\right)k^2, \\ \hat{M}_{k,k} =& \frac{1}{4}\left(-\gamma^2 - 2\left(\gamma^2+1\right)k^2 - 2\left(\gamma^2+1\right)k + 1\right), \\ \hat{M}_{k+1,k} =& \frac{1}{4}\left(\gamma^2 - 1\right)(k+1)^2, \end{aligned} \tag{64}$$

and zero in all other elements of $\hat{M}$. In other words, the differential operator associated with the Truncated Laplace Transform $\mathcal{L}_{1/2,\gamma/2}$ via (38) is a tridiagonal band matrix in the basis of Laguerre Functions.

The second case occurs when $b = 1/2$. The substitution of $a = \frac{1}{2\gamma}, b = 1/2$ into (59) yields

$$\begin{aligned}
\hat{M}_{k-1,k} &= -\frac{\left(\gamma^2 - 1\right) k^2}{4\gamma^2}, \\
\hat{M}_{k,k} &= -\frac{\left(2\left(\gamma^2+1\right)k^2 + 2\left(\gamma^2+1\right)k + \gamma^2 - 1\right)}{4\gamma^2}, \\
\hat{M}_{k+1,k} &= -\frac{\left(\gamma^2 - 1\right)(k+1)^2}{4\gamma^2},
\end{aligned} \tag{65}$$

and zero in all other elements of $\hat{M}$. Again, the differential operator associated with the Truncated Laplace Transform $\mathcal{L}_{1/2\gamma,1/2}$ via (38) is a tridiagonal band matrix in the basis of Laguerre Functions.

3.2.2 The special case: $\mathcal{L}_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}$

An additional special case occurs when $\sqrt{ab} = 1/2$, i.e. in the case of the Truncated Laplace Transform $\mathcal{L}_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}$. Substituting $a = \frac{1}{2\sqrt{\gamma}}, b = \frac{\sqrt{\gamma}}{2}$ into (59) eliminates the first off-diagonal of $\hat{M}$ and yields the matrix $\hat{M}^{(s)}$:

$$\begin{aligned}
\hat{M}^{(s)}_{k-2,k} &= \frac{(\gamma-1)^2(k-1)k}{16\gamma}, \\
\hat{M}^{(s)}_{k,k} &= \frac{\left(-\gamma^2 - 6\gamma - 1\right)k(k+1) - \gamma^2 - 2\gamma + 3}{8\gamma}, \\
\hat{M}^{(s)}_{k+2,k} &= \frac{(\gamma-1)^2(k+1)(k+2)}{16\gamma},
\end{aligned} \tag{66}$$

with zero in all other elements of $\hat{M}^{(s)}$.

3.3 The "standard" form of the Truncated Laplace Transform

As shown in [3], the behavior of the Truncated Laplace Transform $\mathcal{L}_{a,b}$ is determined by the ratio

$$\gamma = b/a. \tag{67}$$

The following lemma summarizes the connections between the SVD of the Truncated Laplace Transform $\mathcal{L}_{a,b}$ and the SVD of the Truncated Laplace Transform $\mathcal{L}_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}$; we will refer to the latter as the *standard form* of the Truncated Laplace Transform.

Lemma 3.5. *Suppose that $0 < a < b < \infty$, and that u_n, v_n and α_n are the $n+1$-th right singular function, left singular function and singular value of $\mathcal{L}_{a,b}$, respectively. Suppose that*

$\tilde{u}_n$, $\tilde{v}_n$ *and* $\tilde{\alpha}_n$ *are the* $n+1$*-th right singular function, left singular function and singular value of the standard form* $\mathcal{L}_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}$*, respectively, with* $\gamma = b/a$.

Then,

$$u_n(t) = \sqrt{\frac{1}{2\sqrt{\gamma}\,a}}\;\tilde{u}_n(t/2\sqrt{\gamma}a), \tag{68}$$

$$v_n(\omega) = \sqrt{2\sqrt{\gamma}\,a}\;\tilde{v}_n(\omega a 2\sqrt{\gamma}), \tag{69}$$

and

$$\alpha_n = \tilde{\alpha_n}. \tag{70}$$

Proof. The identities are readily obtained by the change of variables $t' = t/2a\sqrt{\gamma}$ in (32), the change of variables $\omega' = 2a\sqrt{\gamma}\omega$ in (34), and normalization of the singular functions. $\square$

We denote the differential operator associated with $\mathcal{L}_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}$ via Theorem 2.8 by $\tilde{\hat{D}}_\omega$; this differential operator is specified by the formula

$$\begin{aligned}&\left(\tilde{\hat{D}}_\omega(f)\right)(\omega) = \\ &= -\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left(\omega^2\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}f(\omega)\right) + \frac{\gamma^2+1}{4\gamma}\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\omega^2\frac{\mathrm{d}}{\mathrm{d}\omega}f(\omega)\right) + \left(-\frac{1}{16}\omega^2 + \frac{1}{2\gamma}\right)f(\omega),\end{aligned} \tag{71}$$

which is a special case of formula (36). The following lemma specifies the relation between the eigenvalues of $\tilde{\hat{D}}_\omega$ associated with $\mathcal{L}_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}$ and the eigenvalues of $\hat{D}_\omega$ (defined in (36)) associated with $\mathcal{L}_{a,b}$.

Lemma 3.6. *Suppose that* ${\chi^*}_n$ *is the* $n+1$*-th eigenvalue of the differential operator* $\hat{D}_\omega$ *associated with* $\mathcal{L}_{a,b}$*, and that* $\tilde{\chi^*}_n$ *is the* $n+1$*-th eigenvalue of the differential operator* $\tilde{\hat{D}}_\omega$ *associated with* $\mathcal{L}_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}$*, where* $\gamma = b/a$*. Then,*

$$\chi^*_n = 4\gamma a^2 \tilde{\chi^*}_n = 4ab\tilde{\chi^*}_n. \tag{72}$$

Proof. The relation is obtained from (36) by the change of variables $\omega' = 2a\sqrt{\gamma}\omega$. $\square$

3.4 The operator C_γ

In this discussion, we find it useful to introduce the transform $C_\gamma : L^2\left[\frac{1}{2\sqrt{\gamma}}, \frac{\sqrt{\gamma}}{2}\right] \to L^2[-1,1]$; we define C_γ of a function f by the formula

$$\left(C_\gamma(f)\right)(s) = \gamma^{s/4}\, f\left(\gamma^{s/2}/2\right). \tag{73}$$

Remark 3.7. A simple calculation shows that

$$\int_{-1}^{1} \left(C_\gamma(f)\right)(s) \cdot \left(C_\gamma(g)\right)(s) \mathrm{d}s = \frac{4}{\log \gamma} \int_{\frac{1}{2\sqrt{\gamma}}}^{\frac{\sqrt{\gamma}}{2}} f(t) g(t) \mathrm{d}t. \tag{74}$$

The following lemma provides an expression for the function $C_\gamma\left(\mathcal{L}^*_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}(\Phi_k)\right)$, with Φ_k the Laguerre function defined in (7).

Lemma 3.8. *For any* $1 < \gamma < \infty$,

$$\left(C_\gamma\left(\mathcal{L}^*_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}(\Phi_k)\right)\right)(s) = \gamma^{s/4}\left(\gamma^{s/2}-1\right)^{k}\left(\gamma^{s/2}+1\right)^{-k-1}. \tag{75}$$

Furthermore, the function $C_\gamma\left(\mathcal{L}^*_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}(\Phi_k)\right)$ *is even or odd, depending on the value of* k*:*

$$\left(C_\gamma\left(\mathcal{L}^*_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}(\Phi_k)\right)\right)(s) = (-1)^k\left(C_\gamma\left(\mathcal{L}^*_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}(\Phi_k)\right)\right)(-s). \tag{76}$$

Proof. The identity (75) is obtained by substituting (13) into (73). The symmetry property (76) follows immediately from (75). □

Corollary 3.9. *For* $1 < \gamma < \infty$*, the functions* $\left|\left(C_\gamma\left(\mathcal{L}^*_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}(\Phi_k)\right)\right)(s)\right|$ *decay exponentially as* k *grows, in the following sense*

$$\left|\left(C_\gamma\left(\mathcal{L}^*_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}(\Phi_k)\right)\right)(s)\right| \le \frac{1}{2}\left|\frac{\gamma^{s/2}-1}{\gamma^{s/2}+1}\right|^{k} = \frac{1}{2}\left|1-\frac{2}{\gamma^{s/2}+1}\right|^{k}. \tag{77}$$

Furthermore, if $-1 \le s \le 1$*, then,*

$$\left|\left(C_\gamma\left(\mathcal{L}^*_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}(\Phi_k)\right)\right)(s)\right| \le \frac{1}{2}\left|1-\frac{2}{\gamma^{1/2}+1}\right|^{k}. \tag{78}$$

Proof. By (75),

$$\left|\left(C_\gamma\left(\mathcal{L}^*_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}(\Phi_k)\right)\right)(s)\right| = \frac{\gamma^{s/4}}{\gamma^{s/2}+1}\left|\frac{\gamma^{s/2}-1}{\gamma^{s/2}+1}\right|^{k}, \tag{79}$$

so that for all $1 < \gamma < \infty$ and $s \in \mathbb{R}$,

$$\left|\frac{\gamma^{s/2}-1}{\gamma^{s/2}+1}\right| < 1. \tag{80}$$

The inequality (77) follows immediately from (79) and (80). Equation (78) follows immediately from (77). □

3.4.1 The function $C_\gamma(u_n)$

We introduce the notation U_n for the C_γ of the $n+1$ right singular function of $\mathcal{L}_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}$; U_n is defined by the formula

$$U_n(s) = (C_\gamma(u_n))(s). \tag{81}$$

where u_n is the $n+1$-th right singular function of the operator $\mathcal{L}_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}$ and C_γ is defined in (73).

The following lemma summarizes some of the observations found in [5] and reformulates them using the notation used in this paper.

Lemma 3.10. *The function U_n (defined in (81)) is an eigenfunction of the differential operator $\tilde{\tilde{D}}_s$, defined by the formula*

$$\begin{aligned}\left(\tilde{\tilde{D}}_s \circ f\right)(s) = \left(\log\left(\sqrt{\gamma}\right)\right)^{-2} \frac{\mathrm{d}}{\mathrm{d}s}\left(\gamma^2+1-2\gamma\cosh\left(2s\log\left(\sqrt{\gamma}\right)\right)\right)\frac{\mathrm{d}}{\mathrm{d}s}f(s) \\ -\left(\frac{3}{2}\gamma\cosh\left(2s\log\left(\sqrt{\gamma}\right)\right)+\frac{1}{4}\gamma^2-\frac{7}{4}\right)f(s).\end{aligned} \tag{82}$$

Since the differential operator $\tilde{\tilde{D}}_s$ is symmetric around 0, the function U_n is even or odd in the sense that

$$U_n(s) = (-1)^n U_n(-s). \tag{83}$$

3.5 Decay of the coefficients

Since the left singular function v_n (defined in (27)) is a smooth solutions of a differential equation (specified in (40)), we expect the coefficients η_k^n (defined in (48)) in the expansion of v_n to decay rapidly. In this section we provide an estimate for the actual decay.

Lemma 3.11. *Suppose that v_n is the $n+1$-th left singular function of the operator $\mathcal{L}_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}$ (defined in 24); then*

$$|\eta_k^n| \le \alpha_n^{-1}\frac{\sqrt{2}}{\sqrt{\log\gamma}}\left|1-\frac{2}{1+\sqrt{\gamma}}\right|^k. \tag{84}$$

where η_k^n is defined in (48) and α_n is the $n+1$-th singular value of $\mathcal{L}_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}$.

Proof. By (48) and (27),

$$\eta_k^n = \alpha_n^{-1}\int_0^\infty\left(\int_a^b e^{-\omega t}u_n(t)\mathrm{d}t\right)\Phi_k(\omega)\mathrm{d}\omega. \tag{85}$$

Changing the order of integration and using (25),

$$\eta_k^n = \alpha_n^{-1} \int_{\frac{1}{2\sqrt{\gamma}}}^{\frac{\sqrt{\gamma}}{2}} u_n(t) \left(\left(\mathcal{L}_{\frac{1}{2\sqrt{\gamma}}, \frac{\sqrt{\gamma}}{2}} \right)^* (\Phi_k) \right) (t) \mathrm{d}t. \tag{86}$$

A simple calculation using (74) and (81) shows that

$$\eta_k^n = \alpha_n^{-1} \int_{-1}^{1} U_n(s) \cdot \left(C_\gamma \left(\left(\mathcal{L}_{\frac{1}{2\sqrt{\gamma}}, \frac{\sqrt{\gamma}}{2}} \right)^* (\Phi_k) \right) \right) (s) \mathrm{d}s. \tag{87}$$

By the Cauchy-Schwarz inequality,

$$|\eta_k^n| \leq \alpha_n^{-1} \sqrt{\int_{-1}^{1} (U_n(s))^2 \, \mathrm{d}s} \sqrt{\int_{-1}^{1} \left(C_\gamma \left(\left(\mathcal{L}_{\frac{1}{2\sqrt{\gamma}}, \frac{\sqrt{\gamma}}{2}} \right)^* (\Phi_k) \right) \right)^2 (s) \mathrm{d}s}. \tag{88}$$

Now, since the right singular function u_n is normalized, and using (74) and (81),

$$|\eta_k^n| \leq \alpha_n^{-1} \frac{2}{\sqrt{\log \gamma}} \sqrt{\int_{-1}^{1} \left(C_\gamma \left(\left(\mathcal{L}_{\frac{1}{2\sqrt{\gamma}}, \frac{\sqrt{\gamma}}{2}} \right)^* (\Phi_k) \right) \right)^2 (s) \mathrm{d}s}. \tag{89}$$

Finally, using inequality (78),

$$|\eta_k^n| \leq \alpha_n^{-1} \frac{\sqrt{2}}{\sqrt{\log \gamma}} \left| 1 - \frac{2}{1 + \sqrt{\gamma}} \right|^k. \tag{90}$$

□

Remark 3.12. Lemma 3.11 can be generalized to the following bound for the coefficients of the expansion of the left singular functions of the Truncated Laplace Transform $\mathcal{L}_{a,b}$ in the nonstandard form; the proof is found in our report [11]. The coefficients η_k^n in the expansion of the left singular function v_n of $\mathcal{L}_{a,b}$ are bounded in the sense that

$$|\eta_k^n| \leq \alpha_n^{-1} \sqrt{\frac{2}{\log \gamma}} \left| \frac{\gamma^{s_{max}/2} - 1}{\gamma^{s_{max}/2} + 1} \right|^k, \tag{91}$$

where

$$s_{max} = \max \left(\left| 2 \frac{\log 2a}{\log \gamma} \right|, \left| 2 \frac{\log 2b}{\log \gamma} \right| \right). \tag{92}$$

3.6 Integral of the function u_n

The following lemma describes the relation between the integral $\int_a^b u_n(t)\mathrm{d}t$, where u_n is the right singular function (defined in (27)), and the expansion specified in (43). The expansion (43) is associated with the right singular function u_n via (41) and (42) and it is studied in [12].

Lemma 3.13. *Suppose that u_n be the $n+1$-th right singular function of $\mathcal{L}_{a,b}$, then*

$$\int_a^b u_n(t)\mathrm{d}t = \sqrt{b-a}h_0^n. \tag{93}$$

where h_0^n is the first coefficient in the expansion (42) of the function ψ_n defined in (43) in the basis of Legendre Polynomials.

Proof. By (42),

$$h_0^n = \int_0^1 \psi_n(x)\overline{P_0^*}(x)\mathrm{d}x, \tag{94}$$

Substituting (15) into (94) yields

$$h_0^n = \int_0^1 \psi_n(x)\mathrm{d}x. \tag{95}$$

Now, substituting (41) into (94) with the change of variable $t = a + (b-a)x$ yield (93). $\square$

3.7 A relation between u_n, v_n, and the singular value α_n

The following lemma provides the relation between the coefficient h_0^n (see (42)), the left singular function v_n, and the singular value α_n.

Theorem 3.14. *Suppose that u_n, v_n and α_n are the $n+1$-th right singular function, left singular function and singular value of $\mathcal{L}_{a,b}$ and suppose that h_0^n is the first coefficient in the expansion defined in (43). Then,*

$$\alpha_n = \sqrt{b-a}\,\frac{h_0^n}{v_n(0)} \tag{96}$$

Proof. Evaluating both side of (27) at 0 yields

$$\alpha_n v_n(0) = \left(\mathcal{L}_{a,b}(u_n)\right)(0) = \alpha_n v_n(0) = \int_a^b u_n(t)\mathrm{d}t. \tag{97}$$

Substituting 93 into (97),

$$\alpha_n v_n(0) = \sqrt{b-a}\,h_0^n. \tag{98}$$

$\square$

4 Algorithms

4.1 Computing the left singular function v_n

In this section we introduce an algorithm for the numerical evaluation of $v_n(\omega)$, the $n+1$-th left singular function (defined in (27)) of $\mathcal{L}_{a,b}$ (the operator defined in (24)).

By Theorem 2.10, the function $v_n(\omega)$ is an eigenfunction of the differential operator $\hat{D}_\omega$ defined in (36); by Corollary 3.3, the expansion $\eta^n = (\eta_0^n, \eta_1^n, \ldots)^\intercal$ (defined in (49)) of v_n in the basis of Laguerre functions is an eigenvector of the matrix $\hat{M}$ defined in (59).

Based on Lemma 3.5 and Lemma 3.6, it is sufficient to compute $\tilde{v}_n(\omega)$, the $n+1$-th left singular function of the Truncated Laplace Transform in the standard form $\mathcal{L}_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}$, where $\gamma = b/a$; $v_n(t)$ is computed from $\tilde{v}_n(t)$ using (69). In this special case of the Truncated Laplace Transform, the expansion $\eta^n = (\eta_0^n, \eta_1^n, \ldots)^\intercal$ of $\tilde{v}_n(\omega)$ is an eigenvector of the matrix $\hat{M}^{(s)}$ specified in (66).

Since the matrix $\hat{M}^{(s)}$ is five-diagonal, and is non-zero only in even rows of even columns, and on odd rows of odd columns, an eigenvector of $\hat{M}^{(s)}$ must vanish at either the even or odd positions.

Therefore, for any even $n = 2m$ where $m \geq 0$, equation (49) is reformulated as

$$\tilde{v}_{2m}(\omega) = \sum_{k=0}^{\infty} \eta_k^{even,m} \Phi_{2k}(\omega), \tag{99}$$

where $\eta^{even,m} = (\eta_0^{even,m}, \eta_1^{even,m}, \ldots)^\intercal$ is simply the even numbered elements of the vector η^{2m}; for all integer $k \geq 0$:

$$\eta_{2k}^{2m} = \eta_k^{even,m}. \tag{100}$$

In other words, $\eta^{even,m}$ is the $m+1$-th eigenvector of the tridiagonal matrix $\hat{M}^{(s,even)}$, obtained by removing the odd numbered rows and columns of $\hat{M}^{(s)}$; the non-zero elements of $\hat{M}^{(s,even)}$ are specified by the formula

$$\begin{aligned}
\hat{M}_{k-1,k}^{(s,even)} &= \frac{(\gamma-1)^2(2k-1)k}{8\gamma}, \\
\hat{M}_{k,k}^{(s,even)} &= \frac{2\left(-\gamma^2-6\gamma-1\right)k(2k+1)-\gamma^2-2\gamma+3}{8\gamma}, \\
\hat{M}_{k+1,k}^{(s,even)} &= \frac{(\gamma-1)^2(2k+1)(k+1)}{8\gamma}.
\end{aligned} \tag{101}$$

Similarly, for any odd $n = 2m+1$ where $m \geq 0$, equation (49) is reformulated as

$$\tilde{v}_{2m+1}(\omega) = \sum_{k=0}^{\infty} \eta_k^{odd,m} \Phi_{2k+1}(\omega), \tag{102}$$

where $\eta^{odd,m} = (\eta_0^{odd,m}, \eta_1^{odd,m}, ...)^\intercal$ is simply the odd numbered elements of the vector η^{2m+1}; for all integer $k \geq 0$:

$$\eta_{2k+1}^{2m+1} = \eta_k^{odd,m}. \tag{103}$$

In other words, $\eta^{odd,m}$ is the $m+1$-th eigenvector of the tridiagonal matrix $\hat{M}^{(s,odd)}$, obtained by removing the even numbered rows and columns of $\hat{M}^{(s)}$; the non-zero elements of $\hat{M}^{(s,odd)}$ are specified by the formula

$$\begin{aligned}
\hat{M}_{k-1,k}^{(s,odd)} =& \frac{(\gamma-1)^2 k(2k+1)}{8\gamma}, \\
\hat{M}_{k,k}^{(s,odd)} =& \frac{2\left(-\gamma^2-6\gamma-1\right)(2k+1)(k+1)-\gamma^2-2\gamma+3}{8\gamma}, \\
\hat{M}_{k+1,k}^{(s,odd)} =& \frac{(\gamma-1)^2(k+1)(2k+3)}{8\gamma}.
\end{aligned} \tag{104}$$

Therefore, the algorithm for computing the left singular function v_n of the Truncated Laplace Transform $\mathcal{L}_{a,b}$, where $n = 2m$ is an even number:

Step 1: Compute $\eta^{even,n}$, the $n+1$-th eigenvector of the matrix $\hat{M}^{(s,even)}$, defined in (101).
Step 2: Compute the function $\tilde{v}_n$ from $\eta^{even,n}$, using the expansion specified in (99).
Step 3: Obtain v_n from $\tilde{v}_n$ using (41).

Similarly, the algorithm for computing the left singular function v_n of the Truncated Laplace Transform $\mathcal{L}_{a,b}$, where $n = 2m+1$ is an odd number:

Step 1: Compute $\eta^{odd,n}$, the $n+1$-th eigenvector of the matrix $\hat{M}^{(s,odd)}$, defined in (104).
Step 2: Compute the function $\tilde{v}_n$ from $\eta^{odd,n}$, using the expansion specified in (102).
Step 3: Obtain v_n from $\tilde{v}_n$ using (41).

Remark 4.1. For computations to precision ϵ, the vector $\eta^n = (\eta_0^n, \eta_1^n, ...)^\top$ is truncated at K, such that $|\eta_k^n| \ll \epsilon$ for all $k > K$. By lemma 3.11 the coefficients η_k^n decay rapidly as k grows; since the vectors $\eta^{even,n}$ and $\eta^{odd,n}$, which contain only the even and odd elements of η_k^n, respectively, these vectors are truncated at approximately $K/2$ elements.

The actual position of the last significant coefficient of $\eta^{even,n}$ and $\eta^{odd,n}$, larger in magnitude than ϵ, is given in Figure 6 and Table 4 in Section §5, for several combinations of γ and n.

4.2 Computing the singular value α_n

Theorem 3.14 provides formula (96) for computing the singular value α_n using the function v_n and the first coefficient of the vector h^n (the vector associated with the right singular function u_n via (41) and (42)). The function v_n is computed using the algorithm in Section §4.1, and the vector h^n is the $n+1$-th eigenvector of the matrix M (defined in (44), see [12]).

Remark 4.2. The numerical difficulty in this algorithm is that the first element h_0^n of the vector h^n (see (42)) is of the order of α_n, so that is decays exponentially as n grows. It has been shown in [14] that in some band matrices, such as M, the first element of the eigenvector h^n can be computed to relative precision, not not just to absolute precision. Consequently, if h_0^n is computed as in [14], formula (96) yields α_n with high relative precision.

5 Numerical results

In this section we present results of several numerical experiments. The algorithms for computing the left singular functions v_n and singular values α_n of $\mathcal{L}_{a,b}$ (the operator defined in (24)) were implemented in FORTRAN 77, using double precision arithmetic, and compiled using GFORTRAN.

In Figures 1, 2 and 3 we present examples of left singular functions of the operator $\mathcal{L}_{a,b}$, where $a = 1$ and $b = 1.1$, $b = 10$ and $b = 100000$ respectively.

In Figure 4 and Table 1 we present the singular values α_n of the operator $\mathcal{L}_{a,b}$, for several ratios $\gamma = b/a$; α_n depends only on γ and n (see Lemma 3.5). In table 2 we present several singular values smaller than 10^{-1000}; the Fujitsu compiler with quadruple precision was used in this experiment.

In Figure 5 and Table 3 we present the eigenvalues of the matrix $\hat{M}^{(s)}$ defined in (66).

In Figure 6 and Table 4 we present for several combinations of γ and n the position of the last significant coefficient η^n in the expansion defined in (49), that is larger in magnitude than $\epsilon = 10^{-16}$. In numerical computations, the vectors $\eta^{even,m}$ (defined in (100)) and $\eta^{odd,m}$ (defined in (103)) are truncated at about half this number (see Remark 4.1).

In figure 7 we present the CPU time required for the computation of the expansion of the 101-st right singular function v_{100} of $\mathcal{L}_{1,\gamma}$, for varying γ; The experiment was performed on a ThinkPad X230 laptop with Intel Core i7-3520 CPU and 16GB RAM.

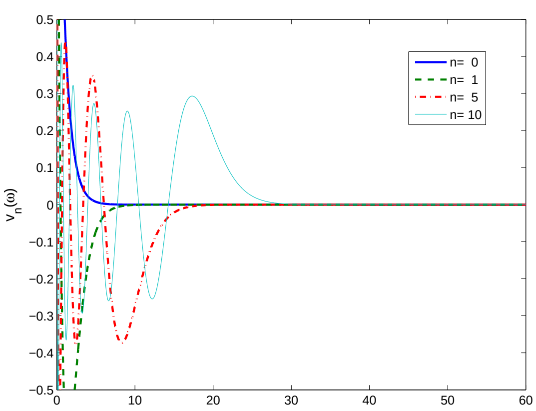

Figure 1: Left singular functions of $\mathcal{L}_{1,1.1}$.

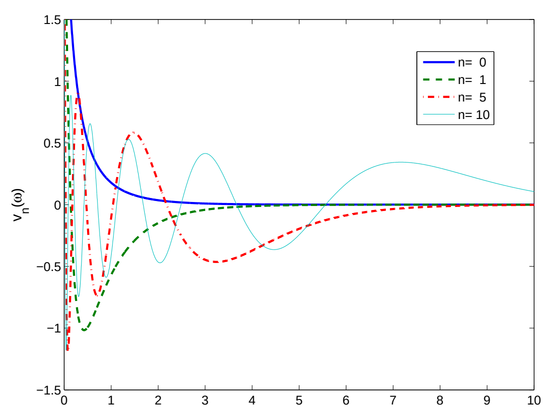

Figure 2: Left singular functions of $\mathcal{L}_{1,10}$.

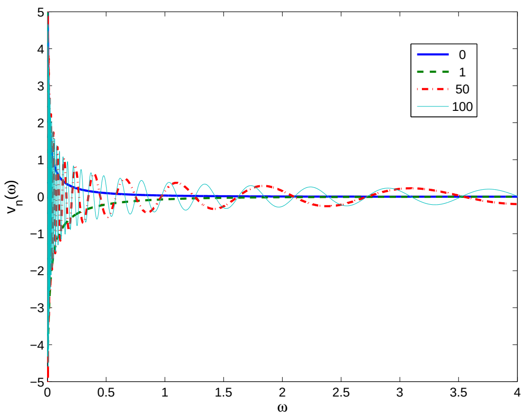

Figure 3: Left singular functions of $\mathcal{L}_{1,100000}$.

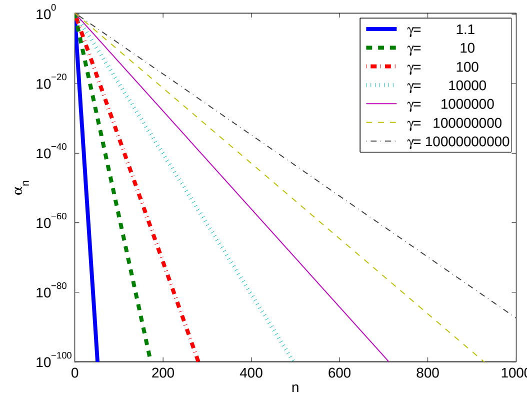

Figure 4: Singular values α_n of $\mathcal{L}_{a,b}$, with $\gamma = b/a$.

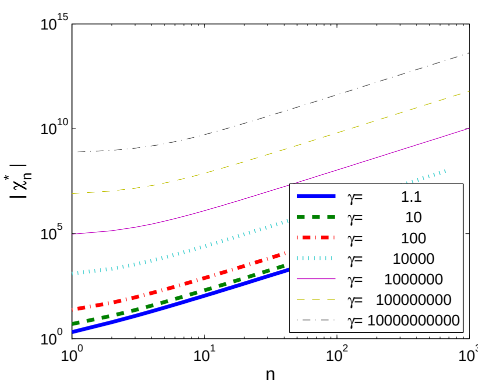

Figure 5: Magnitude of the eigenvalues of the matrix $\hat{M}^{(s)}$ defined in (66).

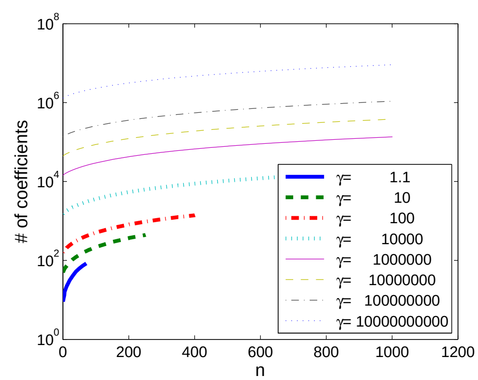

Figure 6: The position of the last significant coefficient larger in magnitude than 10^{-16}.

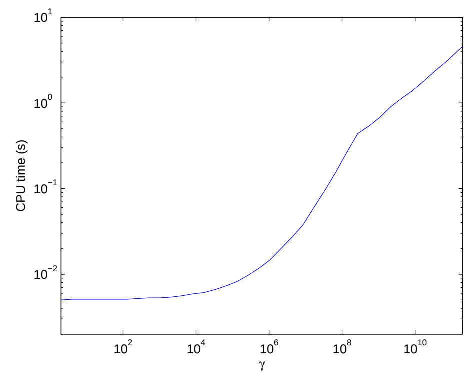

Figure 7: CPU time required for computing the expansion of the 101-st left singular function of $\mathcal{L}_{1,\gamma}$, as a function of γ. The experiment was performed on a ThinkPad X230 laptop with an Intel Core i7-3520 CPU and 16GB RAM.

Table 1: Singular values α_n of $\mathcal{L}_{a,b}$

n	γ =1.0E+01	γ =1.0E+04	γ =1.0E+07	γ =1.0E+10
0	$1.02356E+00$	$1.55687E+00$	$1.67320E+00$	$1.71595E+00$
1	$3.09878E-01$	$1.12288E+00$	$1.43107E+00$	$1.56644E+00$
2	$8.39567E-02$	$7.39927E-01$	$1.14870E+00$	$1.36792E+00$
3	$2.23263E-02$	$4.73173E-01$	$8.92215E-01$	$1.16064E+00$
4	$5.90020E-03$	$2.99697E-01$	$6.82645E-01$	$9.68344E-01$
10	$1.94760E-06$	$1.86336E-02$	$1.28322E-01$	$2.96456E-01$
20	$3.00805E-12$	$1.77967E-04$	$7.70034E-03$	$3.95113E-02$
40	$7.11415E-24$	$1.60942E-08$	$2.74862E-05$	$6.95389E-04$
100	$9.34359E-59$	$1.18179E-20$	$1.24105E-12$	$3.76350E-09$
200	$6.81449E-117$	$7.04566E-41$	$7.08789E-25$	$6.26325E-18$
300		$4.19880E-61$	$4.04637E-37$	$1.04190E-26$
400		$2.50198E-81$	$2.30977E-49$	$1.73305E-35$
500		$1.49081E-101$	$1.31842E-61$	$2.88254E-44$
600		$8.88291E-122$	$7.52539E-74$	$4.79437E-53$
700		$5.29275E-142$	$4.29536E-86$	$7.97413E-62$
800			$2.45170E-98$	$1.32627E-70$
900			$1.39937E-110$	$2.20585E-79$
1000			$7.98724E-123$	$3.66878E-88$

Table 2: Examples of singular values α_n smaller than 10^{-1000}

γ	n	α_n
$1.1E+0$	520	$8.70727E-1002$
$1.0E+1$	1721	$3.66934E-1001$
$1.0E+2$	2797	$5.29961E-1001$
$1.0E+3$	3872	$5.71146E-1001$
$1.0E+4$	4946	$9.44191E-1001$
$1.0E+5$	6021	$8.89748E-1001$

Table 3: Eigenvalues of $\hat{M}^{(s)}$ defined in (66)

n	γ =1.0E+01	γ =1.0E+04	γ =1.0E+07	γ =1.0E+10
0	$-1.37081E+00$	$-7.68147E+02$	$-6.85667E+05$	$-6.58542E+08$
1	$-4.99310E+00$	$-1.24392E+03$	$-8.74386E+05$	$-7.60836E+08$
2	$-1.22170E+01$	$-2.12394E+03$	$-1.20506E+06$	$-9.35829E+08$
3	$-2.30561E+01$	$-3.43924E+03$	$-1.68901E+06$	$-1.18769E+09$
4	$-3.75087E+01$	$-5.19520E+03$	$-2.33192E+06$	$-1.51947E+09$
10	$-2.00102E+02$	$-2.49694E+04$	$-9.57538E+06$	$-5.24213E+09$
20	$-7.60147E+02$	$-9.30877E+04$	$-3.45384E+07$	$-1.80759E+10$
40	$-2.96419E+03$	$-3.61167E+05$	$-1.32782E+08$	$-6.85869E+10$
100	$-1.82480E+04$	$-2.22014E+06$	$-8.14047E+08$	$-4.18853E+11$
200	$-7.26265E+04$	$-8.83424E+06$	$-3.23793E+09$	$-1.66507E+12$
300		$-1.98431E+07$	$-7.27238E+09$	$-3.73934E+12$
400		$-3.52467E+07$	$-1.29174E+10$	$-6.64167E+12$
500		$-5.50450E+07$	$-2.01729E+10$	$-1.03720E+13$
600		$-7.92381E+07$	$-2.90390E+10$	$-1.49305E+13$
700		$-1.07826E+08$	$-3.95157E+10$	$-2.03170E+13$
800			$-5.16029E+10$	$-2.65315E+13$
900			$-6.53007E+10$	$-3.35741E+13$
1000			$-8.06090E+10$	$-4.14447E+13$

Table 4: The position of the last significant coefficient larger in magnitude than 10^{-16}.

n	$\gamma = 1.0E{+}01$	$\gamma = 1.0E{+}04$	$\gamma = 1.0E{+}07$	$\gamma = 1.0E{+}10$
0	50	1502	43890	1282730
1	51	1547	45085	1318363
2	54	1580	45902	1341354
3	57	1611	46583	1359605
4	58	1638	47194	1375474
10	70	1788	50346	1453028
11	71	1813	50831	1464669
20	88	2016	54968	1562968
21	89	2037	55411	1573411
40	122	2438	63464	1762452
41	123	2459	63873	1772049
100	216	3602	86776	2306888
101	219	3621	87149	2315585
200	366	5402	122682	3143252
201	369	5419	123031	3151357
300		7124	156872	3937590
301		7141	157209	3945395
400		8804	190098	4707868
401		8821	190427	4715479
500		10458	222686	5462054
501		10475	223009	5469531
600		12092	254816	6204538
601		12107	255137	6211915
700		13710	286598	6938044
701		13727	286915	6945339
800			318102	7664390
801			318415	7671623
900			349378	8384872
901			349691	8392049
1000			380468	9100436
1001			380777	9107569

6 Conclusions and generalizations

In this paper we introduced effective algorithms for the evaluation of the left singular functions and singular values of the Truncated Laplace Transform $\mathcal{L}_{a,b}$. Together with the algorithms introduced in [12] for the computation of the right singular functions, these algorithms conclude the construction of the SVD of the operators $\mathcal{L}_{a,b}$.

As is evident from Remark 2.3 and the more detailed discussion in [20], the left singular functions of $\mathcal{L}_{a,b}$ are an efficient basis for representing combinations of decaying exponentials whose decay constants are in the interval $[a, b]$.

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